Short-Term Market Risks Implied by Weekly Options: Comments and Extensions

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Abstract

Andersen Fusari and Todorov (AFT) claim that variation in the negative tail risk implied from Weekly options is not spanned by the option market. Using Stochastic Arbitrage, we show that tail risk is adequately represented by the prices of out-of-the money options, but the at-the-money (ATM) Weeklies are way above the prices justified by the dynamics of the S&P 500 index. There is no overlap between option bounds and observed bid-ask spread for most contracts. More than 73% (56%) of ATM calls (puts) are overpriced. In more than 49% (24%) of cross-sections, all ATM call (put) contracts are overpriced. We confirm the overpricing with out-of-sample tests that show high risk-adjusted profits from trading these options in the frictionless market. We attribute the AFT results to the assumptions and distortions imposed on their option data and to market power on the part of the monopolistic liquidity provider licensed by the CBOE.

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I. Introduction

How does one handle the valuation of jump risk in asset pricing models? Anderson, Fusari and Todorov (AFT, 2017) use short term options, known as Weeklies with symbol SPXW, and fit the asset dynamics of the risk neutral frictionless process to the observed option bid-ask spread midpoint (p. 1345, footnote 8). Despite trying many alternative specifications of the risk neutral amplitude, AFT have trouble in identifying models that can adequately fit both right and left tails of the risk neutral distribution; see pp. 1354, 1356, 1358, 1364, etc. They conclude (p. 1370) that "there is a sharp separation between the dynamics of the actual jump risk and its pricing," and mention the need to identify the economic forces that can rationalize their results.

In this paper we turn the AFT results upside down, by presenting empirical evidence that the bid-ask midpoint observed in the option market is an adequate representation of the risk neutral distribution supported by the index return dynamics for the tails of the distribution. By contrast, the option market overprices the highly liquid region around ATM for both calls and puts, with the admissible option prices lying way below the observed bid-ask spread for a large majority of the traded options. We verify that these ATM options are, indeed, overpriced, by out-of-sample tests of suitably designed zero-net-cost portfolios involving the overpriced option, the index and the riskless asset that yield positive risk adjusted expected profits in frictionless trading. These profits decrease extensively if trading is done at the prevailing bid and ask prices. In other words, the AFT results fail not because of "shifts in the pricing of negative tail risk" but because the index dynamics are fatally inconsistent with observed option market prices for a key portion of the support of the risk neutral distribution.

In this paper we present a new theoretical and empirical perspective on the valuation of short term S&P 500 index options, the SPXW "Weeklies" that were introduced by CBOE in 2005 and became increasingly popular, accounting for about 70% of total trading volume of S&P 500 index options by 2021. Our new theoretical perspective is based on recent but little noticed innovations in asset pricing theory, which generalize option valuation, essentially by formalizing the monotonicity of the pricing kernel and combining it with varying asset dynamics of the index.[2] This monotonicity is, in turn, a direct result of the stochastic dominance (SD) approach, which is based on the pairwise comparisons of two otherwise identical risk averse investors, termed index trader (IT) and option trader (OT). The former holds the index and a riskless asset, and the latter adds to the IT holdings a zero net cost option portfolio.

This is the key assumption of stochastic dominance or stochastic arbitrage (SA) option pricing. These IT-OT pairwise comparisons allow the extraction of the risk neutral distribution of the index returns in *frictionless* index and option markets *only* from observed index return data, without relying on option market data. This is possible for almost all cases of market incompleteness that

[2] Kernel monotonicity is discussed further on in this section.

have appeared in practice, including stochastic volatility without and with jumps in returns.[3] The effect of the jumps in all cases is to generate two bounds within which the admissible option prices lie.

Weekly options are a particularly promising field for empirical application of these SA innovations, insofar as the volatility stays approximately constant till option expiration, as assumed in Andersen, Fusari and Todorov (AFT, 2017). This facilitates the extraction of the relevant dynamics of the index, in which the total volatility in our option time series data varies by cross section but stays constant till option maturity for each cross section. Our empirical results are based on parameters estimated from a model that allocates this total volatility to a diffusion component that varies by cross section and to a jump component that stays the same for all cross sections. Once the parameters of these dynamics are extracted from the index return data by the Generalized Method of Moments (GMM), there are closed form expressions that yield the SD bounds for the options in the frictionless markets.

We use these bounds in order to assess the empirical findings of AFT, which were based on a diametrically opposite approach. AFT did not use index return data for their empirical work but extracted jump diffusion parameters of the risk neutral *frictionless* process that by asset pricing theory generates the option values as expectations of the payoffs. The latter holds true *provided* trading is frictionless and the observed option prices are the outcome of perfectly competitive equilibrium. Since the observed option price data is not frictionless but consists of an interval, the bid-ask spread, the midpoint of the spread was used as a proxy for the "correct" price and put-call parity was imposed on the data.

In our case, we compare the frictionless bounds arising out of our SA approach to the observed bid-ask option prices in the corresponding market. On theoretical grounds, it is known that these SA bounds are relatively wide, especially for the out-of-the money (OTM) options. Strikingly, however, this large width does not prevent the bounds from being inconsistent in a major way with the observed option market data. This inconsistency manifests itself by the non-overlap of SD option bounds and bid-ask spread for a very large number of Weekly options, present in almost all the cross sections for both call and put options. Although there are a few option bid-ask spreads that lie below the corresponding SA bounds, in the overwhelming majority of the non-overlaps for both puts and calls the observed spread lies above the theoretical bounds, namely the SD upper bound, implying that the options are overpriced.

To our knowledge, ours is the first empirical option market study to apply a version of asset pricing theory for frictionless markets without interfering with the observed option market data in order to transform it into a suitable frictionless format. Such frictionless data implies put-call parity, which does not hold even approximately in the observed option market quotes. Further, the format of the SD theory allows the computation of individual bounds separately for each option of a given

[3] See Perrakis (2019), Ghanbari, Oancea and Perrakis (2021), and Perrakis and Oancea (2022).

moneyness and maturity. Further, in case of violations of the SD bounds there exist strategies that can exploit such mispriced options, *provided* all trading takes place in a frictionless market. This last assumption is also adopted by most theoretical asset pricing models, that form the basis of the overwhelming majority of empirical index option market studies.

This approach allows us to reconcile the frictionless empirical index option market tests as in AFT for Weeklies, with a different line of empirical work, which uses unadulterated option market data in order to assess whether this data is “correct”, in the sense that it does not allow investors to realize risk adjusted excess profits. In that literature, pioneered by Constantinides, Jackwerth and Perrakis (2009) and extended in Constantinides *et al* (2011), with more recent contributions by Constantinides, Czerwonko and Perrakis (CCP, 2020) and Post and Longarela (PL, 2021), a suitably chosen zero net cost portfolio is added to the index holdings of a generic risk averse investor. That investor, who holds the index and a riskless bond, must not acquire extra risk adjusted expected profit if the options are priced “correctly”, as assessed by the powerful out-of-sample statistical tests of Davidson and Duclos (DD, 2013), for which the null is that the options are correctly priced. This empirical work identified several cases of mispricing, for instance of short futures call options in Constantinides *et al* (2011), or of suitably chosen combinations of short and long option positions in standard S&P 500 index option portfolios in CCP and PL. Such mispricing in the world with frictions implies automatically mispricing in the frictionless world, with cheaper long positions and short positions bringing more money. Since such mispriced portfolios were also present in SPXW options in the CCP empirical work, the AFT results need to be re-examined in the light of these findings.

Our own results also raise several theoretically motivated issues that indicate that the AFT results should be viewed with suspicion. Ever since Constantinides, Jackwerth and Perrakis (2009) showed that equilibrium in the index option market was feasible only if a monotone pricing kernel could pass through the bid-ask spread of most option cross sections, which happened rarely in their data, several other authors have also produced theoretical results that state more or less the same thing. Already in their 1995 article, Jouini and Kallal proved (Theorem 3.2, p. 188) that “correct” frictionless option prices as expected payoffs with a risk neutral index return distribution existed if and only if these prices lay within the observed bid-ask spread in the option market. Similarly, Beare (2011) showed that non-monotone kernels such as the ones fitted to the data in several high-profile studies were inconsistent with frictionless no arbitrage equilibrium. Last but not least, PL showed that a monotone kernel passing through the bid-ask spread was a necessary and *sufficient* condition for efficient pricing in the market with frictions, and a fortiori a sufficient condition for efficient pricing in the frictionless market.

The design and results of our empirical work are tailored to the application of these insights to the SPXW option data. In fact, the results in this paper turn out perversely to support the conclusions of AFT, albeit for entirely different reasons! As they state, “we uncover variation in the negative jump tail risk which is not spanned by market volatility”, which apparently eludes standard asset pricing models. Our own results point out to a very simple explanation for this failure of the standard asset pricing models. For both calls and puts the observed option market prices lie for most cross sections entirely above the SD bounds at the highly liquid zone around the at-the-money (ATM) options, while by contrast for the also liquid deep OTM put options the bid-ask midpoint lies almost always inside the bid-ask spread. A standard frictionless asset pricing equilibrium model applied to this emphatically not frictionless data would have obfuscated this difference that AFT also identified by a different approach, since it would have fitted the same risk neutral distribution to both the OTM values and the overvalued ATM zone.

We also present in this paper a DD-type test that can be applied whenever the observed option market bid or ask prices violate the corresponding upper or lower theoretical option bounds derived on the basis of the extracted index return dynamics. In particular, we present zero net cost strategies involving the violating option, the underlying index tracking fund and the riskless asset, that yield a risk adjusted expected payoff if we assume frictionless trading for both the index and the options. The tests reject overwhelmingly the null of non-dominance in the frictionless world, implying that in such a world the options are overvalued, and the extracted excess returns are not compensations for risk. Further, these excess returns disappear in most cases if the open option positions are closed at the appropriate option market observed ask or bid prices, rather than at the SA bounds, as befits a frictionless market.

Methodologically, our approach has the advantage of relying for empirical estimations and statistical inferences on data extracted from the underlying index, rather than the option market. For the index, the extracted data about the implied index return distributions is relatively "clean", in the sense that it is observable with very little error. By contrast, the extraction of index return data from the option market as in AFT and in most empirical index option research introduces by necessity major approximations, both because of the non-observability of the exact option equilibrium price and the imposition of put-call parity. In our empirical work, option market data is compared to the "clean" estimates of the frictionless prices extracted from the random payoff distributions of the options, in order to extract inferences about their consistence with rational trading in a frictionless world. This consistence hypothesis is decisively rejected by our data.

We conclude that the observed over valuation of the options should be assessed on the basis of equilibrium considerations in the intermediated market, where the bid and ask prices are formed. Although a full modeling of this market is not possible because of the lack of information that allows the derivation of the demand curves for the options, we use the CBOE open-close volume data (liquidity file) for the Weeklies in order to evaluate the risk exposure of the market makers as a group because of their net open positions. Based on information about the officially licensed liquidity providers by the CBOE, we model this market as a monopoly that sets the bid and ask prices for each SPXW option, subject to competition from fringe players that enter with smaller quotes and undercut the official prices. We also include the income from the effective spread on the roundtrip trading for the intermediated volume of the end users.

The evidence from the few aforementioned empirical studies that did take into account the intermediated market is, in fact, suggestive, but leaves several issues unresolved. Constantinides *et al* (2011) discovered anomalous pricing of S&P 500 index futures options, which created risk adjusted profits for the generic index-holding investors. Constantinides, Jackwerth and Perrakis (2009) showed that a monotone kernel capable of pricing the SPX cross sections did not exist for a majority of the observed cross sections, while Constantinides, Czerwonko and Perrakis (2020) found significant risk adjusted profits from suitably selected index option portfolios for the index holding investors. This finding was conformed by Post and Longarela (2021), who added the important clarification that the mispriced option portfolios were directly linked theoretically to the non-existence of a monotone kernel passing through the cross section. Further, that important study gave also a hint about the reason such profitable opportunities were not realized, by noting that there were not sufficient available quotes for the construction of the profitable portfolios to allow for the exploitation of the mispricing by "large" investors. This brings again at the forefront the intermediated market, whose liquidity providers have their own objectives that are not necessarily conducive to efficient pricing according to predetermined standards.

A related and less obvious paradox, that will be tangentially examined in this paper, has to do with the differences between the SPXW options and the SPX options at equal one-week maturities. In Constantinides, Czerwonko and Perrakis (2020) the SPX options were distinctly more mispriced than the SPXW. There were profitable mispriced SPX portfolios in virtually every cross section, unlike SPXW, and the realized risk adjusted profits were larger. As noted above, mispricing in the presence of frictions implies automatically mispricing in the frictionless world. A conjecture that will be verified empirically is that already established market maker positions in the SPX market could not be adjusted without costs to accommodate the trading situation that evolved during the last week of their existence.

In the next section we present the SA theory as it applies to constant volatility jump diffusion asset dynamics in a frictionless world, as well as the design of empirical tests to exploit overpriced in frictionless markets. Section 3 discusses the data and presents the empirical evidence concerning the inconsistence of the SPXW market with the corresponding derived SA bounds. Section 4 presents a theoretical model of a monopolistic market maker quoting bid and ask prices for options as well as the depths of the quotes. It identifies economies of scale and vulnerability to competition from "smaller" players with lower quotes, and conducts several tests based on the liquidity file data. Section 5 concludes.

II. The frictionless index option bounds and their violations

Given the fact that there are closed form expressions for the transformation of the physical or *P*-distribution of the ex-dividend index returns at option maturity, it is vital for our approach to project accurately this distribution. In AFT (eq. (1), p. 1348) the empirical work is carried out in terms of the *Q*-distribution, which is assumed to be stochastic volatility mixed with jumps (SVJ); they also note that trading is supposed to be frictionless. Nonetheless, this distribution is fitted to the midpoint of the observed bid-ask spread of observed options of various maturities including the Weeklies, and put-call parity is imposed in the sample (p. 1345, footnote 8). Last, they note that for such short maturities as in SPXW the volatility can be taken as constant, Since this paper is supposed to be testing the impact of using noisy option data in order to extract the *Q*-dynamics, we shall not follow the AFT approach. Instead, we use *P*-distribution parameter estimates under jump diffusion and assume a volatility that is time-varying between cross sections but constant for each cross section. That volatility is estimated from adjusting the observed VIX index for bias and maturity, as in Constantinides, Czerwonko and Perrakis (2020). As that paper showed, the volatility estimates from the adjusted VIX were excellent forecasts of the ex post observed realized volatility.

An alternative approach would have been the full estimation of the *P*-distribution under SVJ, for which the risk neutralization under SD has been recently developed by Perrakis and Oancea (2022). In fact, there are reasons to believe that the weeklies differ from longer maturity options, as discussed in the empirical implementation in the following sections, and as implied also by the discussion in AFT (p. 1349-1351), which make our adopted approach more suitable for the *P*-parameter estimation. We re-examine this issue in our robustness tests.

Let I_t denote the value of the index at *t*, μ the instantaneous mean assumed greater than the riskless return r, q the (assumed constant) ex-dividend rate, λ the jump intensity, *k* the expected log jump amplitude minus 1, and σ_t the diffusion volatility, which is assumed to stay constant in the interval $[t,T]$ to option expiration *T*. The index dynamics then are given by the following

$$\frac{dI_t}{I_t} = (\mu - q - \lambda k)dt + \sigma_t dW + (j-1)dN \,. \tag{2.1}$$

In what follows we omit for simplicity the dividend rate q, in which case the return on the index is equal to the return of a futures contract F_t on the index that matures at some time after the option maturity, or $\frac{dI_t}{I_t} = \frac{dF_t}{F_t} + \alpha_t dt$, plus a random shock equal to the basis risk. As discussed in Section 4, this random shock varies with the distance from the futures maturity, increases the total volatility, and can be estimated and incorporated into the diffusive volatility.

For the jump amplitude j, we assume that it is a *truncated* lognormal such that $j > j_{\min} \equiv \underline{j}$, which is common to all cross sections, consistent with the assumptions of the jump diffusion SD bounds. With this specification the total variance of the ex-dividend index return till option expiration is given by the following expression, which is observable at every cross section and equal to the bias-adjusted VIX

$$Var\left[\ln\frac{I_T}{I_t}\Big| j > \underline{j}\right] = \left[\sigma_t^2 + \lambda\left\{\left(E\left[\ln(j)\middle| j > \underline{j}\right]\right)^2 + \left(\text{var}\left[\ln(j)\middle| j > \underline{j}\right]\right)\right\}\right](T-t) \,. \tag{2.2}$$

The constant instantaneous mean μ can easily be varied or set proportional to variance or to volatility, with very little effect on the results. The parameter $\underline{j}$ is important and affects the option bounds that widen when it is reduced. In our base case it is set at 0.8, implying a very generous 20% drop in a single jump. At the limit, as $\underline{j} \to 0$ the amplitude distribution becomes a full lognormal and the SD upper bound (2.3) attains its highest value and must be modified, since it is no longer risk neutral. This case is unrealistic and is discussed briefly in the empirical section and the robustness checks.

Given the estimated parameters of the time-varying but constant volatility by cross section jump diffusion model, the SD option bounds are payoff expectations with the following boundary *Q*-distributions.[4] For the upper bound

$$\frac{dI_t}{I_t} = \left(r - \left(\lambda + \lambda_{Ut}\right)k^U\right)dt + \sigma_t dW_t^Q + \left(j_t^U - 1\right)dN_t^Q, \tag{2.3}$$

where the upper bound risk-neutral jump intensity is $\lambda^U = \lambda + \lambda_{Ut}$ and

$$\lambda_{Ut} = -\frac{\mu - r}{\underline{j} - 1}$$

and j_t^U is a mixture of jumps with intensity $\lambda + \lambda_{Ut}$ and distribution and mean

$$j_t^U = \begin{cases} j & \text{with probability } \frac{\lambda}{\lambda+\lambda_{Ut}} \\ \underline{j} & \text{with probability } \frac{\lambda_{Ut}}{\lambda+\lambda_{Ut}} \end{cases}.$$

$$E\left[j_t^U - 1\right] = k^U = \left(\frac{\lambda}{\lambda + \lambda_{Ut}}\right)k + \left(\frac{\lambda_{Ut}}{\lambda + \lambda_{Ut}}\right)\left(\underline{j} - 1\right)$$

For the lower bound we have

$$\frac{dI_t}{I_t} = \left[r - \lambda k^L\right]dt + \sigma_t dW_t^Q + (j_t^L - 1)dN_t^Q \tag{2.4}$$

where the lower bound's jump intensity remains the same, $\lambda^L = \lambda$, and j_L^t is absolute jump size with the truncated distribution $j \mid j \le \overline{j}_t$.

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[4] See Perrakis (2019, pp. 45-61).

The mean of the relative jump size, k^L, and the value of truncation boundary $\overline{j}_t$ can be obtained by solving the equations

$$\begin{aligned} \mu - r &= \lambda k - \lambda k^L \\ k^L &= E(j-1 \mid j \leq \overline{j}_t) \end{aligned}.$$

We noted in the introduction that the derived bounds on the basis of (2.3) and (2.4) can be applied individually to each traded option and do not require consistency in their applications with respect to moneyness and maturity. Further, the frictionless equilibrium value of an option can vary between options with different degrees of moneyness in a given cross section even if the options are priced "correctly". Indeed, our setup allows clientele effects that make the demand for options differ between degrees of moneyness and allow inferences with respect to differential behavior of at-the-money (ATM) and OTM options similar to the ones in AFT, as discussed in our empirical Section 4.

An equilibrium price of an option consistent with the SD bounds (2.3)-(2.4) can be derived by using the device of a representative investor of the constant relative risk aversion (CRRA) type and with restrictions on the index units in her optimal portfolio holdings. This was shown in Ghanbari, Oancea and Perrakis (GOP, 2021), who also showed that it is possible to decompose the risk premium allocation between diffusive and jump risk. Note that in the traditional simultaneous equilibrium models of the index and option markets such as Bates (1991) and Amin and Ng (1993) the relative risk aversion (RRA) value is unique for all options and all levels of moneyness and maturity and cannot handle varying volatility. Here, by contrast, the RRA values corresponding to (2.3)-(2.4) and a given optimal portfolio lie within a given interval, for which the maximum corresponds to the upper bound and the lower bound is still risk neutral but lies below the Merton (1976) value that corresponds to unsystematic jump risk. In-between the Merton value and the upper bound the equilibrium corresponds to an RRA value that divides the premium allocation between the two sources of risk. This allocation is also discussed in the following section.

A major feature of the frictionless SD jump diffusion bounds is the size of the relative spread of the two SD bounds (2.3)-(2.4) as a function of the degree of moneyness. It was shown in GOP (2021, p. 255), that the spread varies inversely with moneyness, with the highly liquid deep OTM options having the widest relative spreads, for both calls and puts. This property plays a major role in our interpretation of the empirical results in the following section.

Next, we derive strategies that exploit the violation of the bounds by an observed option price in a given cross section, assuming always that trading is frictionless. These strategies would perforce be applied to a discretized version of the ex-dividend dynamics (2.1)-(2.4), as shown below for (2.1), setting the return $z_{t+\Delta t} = \dfrac{I_{t+\Delta t} - I_t}{I_t}$

$$z_{t+\Delta t}=\begin{cases}[\mu-\lambda k]\Delta t+\sigma_t\varepsilon\sqrt{\Delta t} & \text{with probability } (1-\lambda\Delta t)\\ [\mu-\lambda k]\Delta t+\sigma_t\varepsilon\sqrt{\Delta t}+(j-1) & \text{with probability } (\lambda\Delta t)\end{cases}\quad . \tag{2.5}$$

The strategies are based on the derivation of the SD bounds, shown in the proof of Lemma 1 in Perrakis (2019, pp. 23-27). The proof is provided in a single period model-free format for any distribution $\Pi_t(z_{t+\Delta t})$ of the return, and then applied recursively to the sequence of time periods $\tau\in[t,T-1]$ till option maturity. By definition, the derived upper bound for a call option is such that shorting a call and allocating the proceeds to the index and the riskless asset in a dynamically rebalanced portfolio yields a payoff at expiration whose risk adjusted expectation is exactly equal to 0. Conversely, if one shorts a call at a higher value than the SD upper bound the risk-adjusted payoff in a frictionless market is positive and constitutes an anomaly. Setting $Min\{z_{t+\Delta t}\}\equiv z_1$, $e^{r\Delta t}=R$ and $E_t[z_{t+\Delta t}]\equiv\hat{z}_t$, the *Q*-distribution for the upper bound $U_t(z_{t+\Delta t})$ for both call and put options is shown in equation (2.3) of Perrakis (2019, p. 24) as

$$U_t(z_{t+\Delta t})=\begin{pmatrix}\Pi_t(z_{t+\Delta t}) \text{ with probability } \dfrac{R-(1+z_1)}{\hat{z}_t-z_1}\\ 1_{z_{t+\Delta t}=z_1} \quad \text{with probability } \dfrac{1+\hat{z}_t-R}{\hat{z}_t-z_1}\end{pmatrix}. \tag{2.6}$$

Suppose we observe at time *t* a call option in a cross section of a given maturity, whose bid price lies above the SD upper bound, namely the payoff expectation with (2.6), or $C_{bt}>\bar{C}(I_t,T)$. The strategy consists in shorting one call per unit index and allocating β_t and $1-\beta_t$ in the riskless bond and the index, respectively. This position is closed at time $t+\Delta t$ at a price equal to the call upper bound $\bar{C}(I_{t+\Delta t},T)$, which is derived from the Q-dynamics in (2.6) for all risk averse traders. The allocation β_t is chosen so that at the lowest value of the return $z_{t+\Delta t}$ the closed short option position at the upper bound $\bar{C}(I_{t+\Delta t},T)$ yields a zero payoff for the portfolio. Further, the allocation is rebalanced at any point $\tau\in(t,T-1]$, with β_τ similarly chosen so that at the lowest value of the return $z_{\tau+\Delta\tau}$ the portfolio payoff at the upper bound $\bar{C}(I_{\tau+\Delta\tau},T)$ would be equal to 0. At option maturity *T* the cumulated allocation would be equal to $C_{bt}\left[\prod_{\zeta=t}^{\zeta=T}R^{\Delta\zeta}\beta_\zeta+\prod_{\zeta=t}^{\zeta=T}(1-\beta_\zeta)(1+z_{\zeta+\Delta\zeta})\right]$, from which we need to subtract the proceeds of the closed short call $\bar{C}(I_T,K,T)=(I_T-K)^+$. Thus, the excess returns at *T* would be

$$C_{bt}\left[\prod_{\zeta=t}^{\zeta=T}R^{\Delta\zeta}\beta_\zeta+\prod_{\zeta=t}^{\zeta=T}(1-\beta_\zeta)(1+z_{\zeta+\Delta\zeta})\right]-(I_T-K)^+ . \tag{2.7}$$

A similar procedure can be carried out at any $\tau \in (t, T-1]$ whenever we observe $C_{b\tau} > \overline{C}(I_\tau, T)$, although this will not be included in our tests.

For an overpriced put option, we write a put at its bid price P_{bt} if $P_{bt} > \overline{P}(I_t, T)$, short $\beta_t I_t - P_{bt}$ of index with $\beta_t < 1$, and invest $\beta_t I_t$ in the riskless asset. The payoff at $t+\Delta t$ is $\beta_t I_t R - [\beta_t I_t - P_{bt}](1+z_{t+\Delta t}) - P(I_t(1+z_{t+\Delta t}), T)$, whose lowest value is when $P(I_t(1+z_{t+\Delta t}), T) = \overline{P}(I_t(1+z_{t+\Delta t}), T)$. This payoff is clearly increasing in its put bid price P_{bt} for every β_t. At the lowest value $z_{t+\Delta t} = z_1$ the payoff should be nonnegative, implying that

$$\beta I_t R - (\beta S_t - P_{bt})(1+z_1)] - \overline{P}(I_t(1+z_1)) \geq 0 \Rightarrow \beta I_t(R-(1+z_1)) + P_{bt}(1+z_1) - \overline{P}(I_t(1+z_1) \geq 0,$$

from which we get $\beta_t \geq \frac{\overline{P}(I_t(1+z_1)) - P_{bt}(1+z_1)}{I_t[R-(1+z_1)]} \equiv \beta_t^*$. Setting β_t at this value, at $t+\Delta t$ the expectation of the payoff is $\beta_t^* I_t R - E_t[(\beta_t^* I_t - P_{bt})(1+z_{t+\Delta t}) - \overline{P}(I_t(1+z_{t+\Delta t}))]$. This expectation should be 0 at $P_{bt} = \overline{P}(I_t(1+z_{t+\Delta t}), T)$, which is verified using (2.6). Hence, it is positive for $P_{bt} > \overline{P}$, implying that we have a positive payoff from the strategy when we close the position at $\overline{P}(I_{t+\Delta t}, T)$. These short put and short index positions are readjusted for every $\tau \in (t, T-1]$ by varying the parameter β_τ and setting β_τ^* at a value for which the payoff is 0 when $z_{\tau+\Delta\tau} = z_1$, namely $\beta_\tau^* = \frac{\overline{P}(I_\tau(1+z_1)) - P_{bt}(1+z_1)}{I_\tau[R-(1+z_1)]}$. At maturity the cumulated payoffs from the short put position are

$$P_{bt} \prod_{\zeta=t}^{\zeta=T-1} ((1+z_{\zeta+\Delta\zeta}) + \sum_{\tau=t}^{\tau=T-1} [\beta_\tau^* I_\tau \{R^{T-\tau} - \prod_{\zeta=\tau}^{\zeta=T-1} (1+z_{\zeta+\Delta\zeta})\}] - (K - I_T)^+ . \tag{2.8}$$

These strategies for both overpriced calls and overpriced puts in the frictionless world are clearly executable, and their outcomes are observable in the real world. Equivalent, equally executable and observable strategies exist whenever the observed ask prices of the options lie below the corresponding SD lower bounds which, however, occur much less frequently and will not become part of our empirical work. Instead, we verify whether these zero net cost strategies that exploited the overpricing of call and put options in the frictionless world survive the out-of-sample SD tests. For this we need to prove that when their realized payoffs per unit index are added to the terminal holdings of a generic risk averse investor who allocates optimally her wealth between the index and the riskless asset the combined portfolio is unanimously preferred irrespective of the investor's risk aversion. The design of this empirical work is presented in the following section, while Section 4 presents the results and also compares the observed bid and ask option prices to the only existing theoretical standards for a market with frictions, the Constantinides-Perrakis (2002) option bounds under proportional transaction costs.

III. Data and Empirical Design

Weekly S&P 500 index options (SPXW) are similar to standard monthly options except they have a shorter life span, are PM-settled on their expiration date, and are expiring every day of the week. They are typically listed several weeks in advance. Since launch, SPX Weekly options have grown to become one of CBOE's most-actively traded products. A total of 345 million S&P 500 index option contracts were traded in 2021, with an average daily volume (ADV) of 1.4 million contracts. Among them, there were approximately 247 million SPXW contracts in 2021, with an ADV of more than 981,000 contracts, accounting for nearly 72 percent of total SPX trading volume.[5] In this study we only focus on the End-of-Week SPXW options, which are issued on a Thursday and mature on the next Friday.

AFT's empirical objectives were to use the Weeklies in order "to extract information about the current state of the risk-neutral dynamics (volatility and conditional tail probabilities)" of the S&P 500 index. In their conclusions the authors note (p. 1370) that "there is a sharp separation between the dynamics of the actual jump risk and its pricing", which they attribute to nonlinearities in the pricing kernel. As we shall see, a more likely explanation is the wider spreads on theoretical grounds of OTM put options under SD. In turn, these wide spreads are due to the fact that under SD the OT investor payoffs must lie above those of IT at the left tail of the returns distribution.

Our SD approach allows a more accurate assessment of this apparent disconnect between the risk of jump occurrence imbedded into the index return dynamics, and the way it is reflected in the observed prices in the option market. There are several points that are relevant in order to explain that disconnect. The most important is the fact that the pricing of the jump diffusion risk, which is measured in AFT, as in most empirical research on index options, by fitting the risk neutral Q-distribution to the observed option bid-ask midpoint, is only applicable to frictionless trading. In the presence of frictions, the few available theoretical results indicate that the valuation of options is done by payoff expectations with distributions that are not necessarily risk neutral.[6]

Equations (2.3)-(2.4), both applications to jump diffusion index dynamics of the model free risk neutral Q-dynamics whose only assumption is the monotonicity of the pricing kernel, effectively separate the pricing of volatility and jump risk from observed option market data. Now the disconnect manifests itself by the inconsistency of the option prices evaluated by the risk neutral Q-dynamics with the observed bid and ask prices in the option market. This market is an intermediated market, where liquidity-providing market maker(s) interact with end users, whose net demands for call and put options are different. The availability of data from the CBOE's liquidity files for Weeklies, allows us to observe the net positions of the liquidity provider separately for each type of option and degree of moneyness. The theoretical analysis of this

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[5] https://ir.cboe.com/news-and-events/2022/04-13-2022/cboe-add-tuesday-and-thursday-expirations-spx-weeklys-options

[6] See the discussion in Perrakis (2019, Ch. 3).

intermediated market and its linkage with the observed bid and ask option quotes are done in the following section.

For the empirical estimation of the index returns' *P*-dynamics we use as inputs observed ex-dividend daily returns over the period January 3, 1963 to December 31, 2019. During this period the average annualized return is 6.92%, the standard deviation of returns is 15.99%, and their skewness and kurtosis are -1.01 and 29.91 respectively. In our robustness checks we also use daily returns from January 2, 1980 to December 31, 2019. For the index price we use the price provided by the CBOE reporting system.[7] For the dividend yield we use daily cash payouts obtained from Standard and Poor. For the interest rate, we use the 3-month constant maturity T-bill rate obtained from the Federal Reserve Economic Data.

As noted in the previous section, we use as total variance of the index returns in each cross section the observed VIX index adjusted for its bias. The latter is equal to the mean forecast difference between the VIX and the realized volatility from 1986 to the current date. Both the VIX and the realized volatility of daily returns are measured in one-week intervals without overlap, with the latter quantity defined as the square root of 252 times the mean squared daily return. The amount by which the VIX exceeds the realized volatility (the negative volatility risk premium) is shown in Perrakis (2019, p. 216, Figure 6.1). The figure indicates that this estimate of the average volatility risk premium is relatively stable over time at about 4.5%–4.8%. In our robustness checks, we also consider alternative specifications of the VIX bias with very little impact on our results.

From the time series of the adjusted VIX observations we use expression (2.2) in order to extract the diffusion volatility σ_t^2 for each option cross section, given the parameters of the jump component that are common to all cross sections. As noted, we use for the jump a constant intensity λ and for the amplitude a truncated lognormal at $\underline{j} = 0.8$, whose moments are shown in Ghanbari, Oancea and Perrakis (2021, Appendix B). Applying the method of moments to the ex-dividend returns, we extract the following parameter values: $\mu = 8.20,\ \lambda = 0.2408,\ E\left[\ln(j)\middle| j > \underline{j}\right] = -0.0294,\ \text{var}\left[\ln(j)\middle| j > \underline{j}\right] = 0.10$. From these parameters we get the following risk neutral jump parameters for the two bounds: $\lambda^U = \lambda + \lambda_{Ut} = 0.3307,\ k^U = -0.0679,\ k^L = -0.0987$. We also generate the time series of diffusion volatilities σ_t for all cross sections, whose range extends from a low of 3.15% to a high of 73.63%. The statistical properties of our parameter estimates are in Table I of our online appendix. From these *P*-parameters we estimate the upper and lower bounds of the options in each cross section for every degree of moneyness that appears in the option data set.

[7] In studies that used earlier data the intraday index price was found from the cost-of-carry relationship between the cash index and its futures price, due to poor reporting of the cash index. As of 2007 the quality of this reporting has significantly improved and there is no reason to use futures.

The estimated bounds for the SPXW options are compared to the observed quotes of these options in 380 cross sections between 2009/11/06 and 2019/07/19. Unlike AFT, who use end-of-day data from OptionMetrics, we extract the option quotes from the CBOE tape with intraday quotes. We apply the usual moneyness and liquidity filters, which are very similar to the ones used in Constantinides, Czerwonko and Perrakis (2020). Specifically, after eliminating obvious data errors we filter the data by imposing put–call parity and convexity with respect to the strike price under transaction costs in the index and bid–ask prices of options. We conservatively use 10 basis points as a one-way transactions cost. We include call prices with bid prices of at least $0.15. Last, we apply liquidity filters based on volume to make sure that only options that can be traded under realistic conditions enter our data set.

Table 1 shows the characteristics of the options that pass our filters, taking into account that put options are much more liquid than call options. Overall, there are 380 cross sections, with a total range of degrees of moneyness strike over index K/I_t from 0.57 to 1.20 that qualify for inclusion in the results. It is immediately obvious from these results that put-call parity is very hard to justify in attempting to extract a frictionless Q-distribution from these data, as was already pointed out in Perrakis (2022). This fact is on its own sufficient justification for the separate modeling of the frictionless option market that forms the core of this paper. There are about twice as many put options traded as there are calls, no matter what metric is used to measure trading activity, open interest, volume or number of contracts.[8] There are also major differences in the width of the liquid moneyness range, which in terms of volume is for calls $K/I_t \in [0.98, 1.20]$ and $K/I_t \in [0.57, 1.02]$ for puts. In terms of the width of the observed bid-ask spread as a proportion of the quote midpoint, the quotes are tightest in the few liquid in-the-money (ITM) buckets, remain tight in the highly liquid ATM zone of $K/I_t \in [0.98, 1.02]$, and escalate dramatically at the also liquid OTM zones for both calls and puts. The table also shows the relative spreads of the SD bounds in each moneyness category. Last, the table shows the net positions of the end users, separately for calls and puts, which are negative for calls and mostly positive for puts.

[Table 1 about here]

In our empirical work these spreads will be confronted with the theoretically estimated SD bounds. As expected from the examples presented in GOP, the bounds are quite wide at OTM, reflecting the uncertainty associated with the estimation of the frictionless equilibrium price.[9] For all the width of the SD bounds the fact is that they are still in many and even most cases far below the observed data in the option market, as shown in our following sets of results.

As noted in Perrakis and Oancea (2022), each value within the SD bounds reflects differing allocations of the total risk premium between diffusive volatility and jump risk, with the upper

[8] It is possible that there is missing information about call traded volume in the older part of our data set.
[9] Note that our moneyness metric is the conventional strike over current asset price, not the one used by AFT (p. 1345) that normalizes it by dividing by maturity and ATM implied volatility.

(lower) bound corresponding to the entire premium allocated to the diffusive (jump) risk. In-between the bounds the risk allocation depends on the aggregation of demands and can be assessed for traders with CRRA preferences with standardized optimal IT portfolio holdings. For such traders the aggregation depends on the value φ of the RRA and stays constant for any given cross section at every time point along the path to option maturity, since the returns for every time partition are independent and identically distributed (iid). The lower SD bound corresponds to a value $\varphi < 0$,[10] while for the upper bound there exists a cross section-specific value $\varphi_t^{\max}$ that shows the highest RRA value for an investor who holds one index unit and participates in the option market. For such investors with $\varphi \in (0, \varphi_t^{\max})$ the allocation was shown under the more general stochastic volatility with jumps case in Proposition 4 and relation (4.9) of Perrakis and Oancea (2022), reproduced below in our notation.

$$\varphi_{t,T}^{\max} = Max\left(\varphi \left| \begin{array}{l} \mu - r_t = \varphi\sigma_t^2 + \lambda\kappa - \lambda^Q\kappa^Q, \\ \lambda^Q = \lambda E_t(j^{-\varphi}),\ \kappa^Q = \kappa E_t[\dfrac{(j-1)j^{-\varphi}}{E_t(j^{-\varphi})}] \end{array} \right. \right) \quad . \tag{3.1}$$

Note that the actual equilibrium need not involve only CRRA investors with $\varphi \in (0, \varphi_t^{\max})$, since there is no reason to exclude other types of investors in the aggregation of option positions. In a frictionless world an option value that lies above its SD upper bound is a profitable riskless trading opportunity for all risk averse investors.

The second and most important building block of our empirical work is the confrontation between the observed SPXW prices of Table 1 and the theoretical SD bounds extracted from the *Q*-dynamics (2.6). This is shown in Table 2, which shows in two different panels respectively for calls and for puts the number of cross sections and the corresponding proportions by degrees of moneyness of the observed cases where the bid-ask midpoint lies within the SD bounds and the cases where there is no overlap between bounds and observed bid-ask spread. Since almost all such inconsistencies occur because of overpricing of the options in the market, they imply that the observed bid prices lie entirely above the SD upper bound.

[Table 2 about here]

The results are striking in their uniformity in terms of what they say for both calls and puts, in spite of the vastly differing characteristics of the two option markets. For the 0.98-1.0 and 1.0 to 1.02 moneyness categories, there are 68% and 75% respectively of the cross sections where more than half of the call options quotes lie entirely above the SD upper bound; the corresponding values for the puts are 81% and 56% of cross sections with more than 50% of overpriced quotes. Note that

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[10] The SD lower bound lies below but very close to the Merton (1976) bound that corresponds to risk neutrality on the basis of the fully diversifiable jump risk in a general equilibrium model.

there is a major asymmetry here between calls and puts, since these moneyness ranges contains 53.12% of total volume (7,567) for calls but only 18.15% of total put volume (38,219). It is not, therefore, possible to extract frictionless option market asset dynamics from these quotes for either calls or puts, let alone for both if we impose put-call parity.

Such frictionless dynamics, on the other hand, can be extracted from the deep OTM put range of 0.86-0.96, which contains 46.41% of total trading volume. The overwhelming majority of the put option quote midpoint, the proxy for the assumed frictionless option price, does indeed lie within the quotes and is priced "correctly", namely consistently with the *P*-dynamics. In other words, the deep OTM put options are consistent with the aforementioned Jouini-Kallal (1995) theorem that a correct frictionless price must lie within the bid-ask spread, whereas the options in the ATM range, both calls and puts, are mispriced. The results for the OTM calls, in the moneyness ranges 1.02-1.04 and 1.04-1.06, are mixed, with the midpoint lying within the bounds for only 44% and 61% of the cases respectively, as shown in the first row of Table 2.

We conclude that a frictionless set of option prices that corresponds to conventional criteria of efficient market equilibrium and is even minimally consistent with the estimated *P*-dynamics cannot be extracted from the observed market prices of SPXW options. This was, perhaps, to be expected: recall that in the CCP results the zero net cost portfolios of SPXW options had generated significant excess returns that withstood the stringent out-of-sample DD test. As we noted in the introduction, if these portfolios were mispriced in the market with frictions, they would be a fortiori mispriced in a frictionless market.

The final empirical results of this section are mispricing tests of the observed equilibrium violations in the close to ATM zone in Tables 3 and 4 in a world without frictions. For each call option cross section with n observed non overlaps such as $C_{bt}^{i}(I_t, K_i, T) > \bar{C}_i(I_t, K_i, T), \;\; i = 1, ..., n$ we form a portfolio of the weighted sums of the differences $\sum_{1}^{n} w_i(C_{bt}^{i} - \bar{C}_i)$, where the weights are positive and proportional to the deviations of the bid prices from the calls' upper bounds, with weights summing to one and maximizing the total mispricing. A similar procedure is applied to the violating put options in every cross section. At option maturity each one of these violating options will contribute to the mispricing an amount equal to (2.7) for calls and (2.8) for puts, and the weighted average of these amounts with the optimal weights chosen at t is the cross section's contribution to mispricing. These are shown in Figure 1 in terms of one-day log returns of the call and put portfolios, as well as the IT and corresponding OT portfolios. Figure 2 shows the same information in terms of cumulative one-day returns over the period of the data.

[Figure 1 about here]

[Figure 2 about here]

These figures show very clearly the major enhancement of the IT returns from the zero net cost option portfolios and set the stage for the formal DD tests. These were first used in Constantinides *et al* (2011), and verify whether the sum of the index returns (the IT holdings), and the suitably standardized time series of returns from zero cost option portfolios over all violating cross sections (namely the OT holdings), dominates the index return series. The tests are particularly convincing since the null is non-dominance. The results of these tests are shown in Table 3.

[Table 3 about here]

[Table 4 about here]

Table 3 shows clearly that the options included in the tests, both calls and puts, are mispriced in a frictionless market. The DD tests reject unequivocally the non-dominance tests in all cases, which implies that the realized excess return of the OT portfolios is higher than that of IT, while the realized volatility is similarly lower than that of IT, as shown in the next Table 4. In other words, the observed calls whose bid-ask spread lies above the SD bounds are an easily executable SA opportunity, provided trading is frictionless.

To verify whether these frictionless SA opportunities are also profitable SA opportunities in the presence of frictions it suffices to close the short option positions at the ask prices of the options. For this we replace the last terms in the expressions (2.7) and (2.8) by $R^{T-t}C^{i}_{at}(I_t, K_i, T)$ and $R^{T-t}P^{i}_{at}(I_t, K_i, T)$, respectively, which are obviously much higher than the risk neutral expectations of the option payoffs evaluated at the SD upper bound at time $t+\Delta t$. The results are shown in the second column of Table 4. It is clear for calls that the excess OT profits disappear and there is no need for a formal DD test. The results are more nuanced for puts, where some excess return still exists, albeit much reduced with respect to the frictionless case. We conclude that the mispricing of the options in the frictionless world does not carry over in the presence of frictions. The resolution of these contradictory findings obviously depends on the modelling of the option bid and ask prices with the *Q*-dynamics (2.6) of the upper bound, which are examined in the next section.

IV. Bid and Ask Option Quotes for SPXW

In June 2022 the CBOE web site indicated that “A "Designated Primary Market-Maker" or "DPM" is a Trading Permit Holder organization that is approved by the Exchange to function in allocated securities as a Market-Maker on the trading floor.”[11] Further, it also indicated that “Only one DPM may be appointed per class”, and that “DPMs may apply to be allocated new listings (or classes being reallocated from another DPM organization) through an exchange sponsored solicitation process.” Last, the information provided in their circular “34-93955-ex5.Liquidity providers’

[11] See https://www.cboe.com/us/options/trading/liquidity_providers/.

quoting standards" implies that the monopolistic DPM was paying licensing fees to the exchange, although the amount and calculation method were not given. There was also no information on the selection criteria in the CBOE's Rule Book, except in general terms. In what follows we shall assume that there is only one liquidity provider for the entire SPXW class, who has sole access to the incoming orders from end users, which she condenses into an order book that shows the available bid and ask quotes together with their depths for each degree of moneyness. In other words, the DPM is the only market participant who has accurate information on the net demand for each traded Weekly option.

We assume, following Constantinides and Lian (2021), that the markets for put and call options are segmented and model the DPM behavior for calls, with the one for puts following similar mechanisms. Denote by C_f the frictionless upper bound, and let (C_b, C_a) the bid-ask pair for call options quoted by the monopolistic market maker. Assume that the total end user demand for a given option is $N_s = D_s(C_b, C_a, t)$ for short and $N_l = D_l(C_b, C_a, t)$ for long, where the first is increasing in C_b and the second is decreasing in C_a. Assume also, as is reasonable, that $\frac{\partial D_S}{\partial C_a} \geq 0$ and $\frac{\partial D_l}{\partial C_b} \leq 0$. The total net position is $N_s - N_l \equiv N_e > 0$, where N_e is the net exposure. Then the monopolistic market maker is fully hedged in $N_l = D_l(C_b, C_a, t)$, with exposure equal to N_e, which is increasing in both C_b and C_a.

We assume, as is plausible, that the DPM in the world with frictions holds wealth in the form of cash and hedges her uncovered option positions with S&P 500 index futures. In other words, she acts as an IT investor holding a portfolio with x_t in the riskless asset and a starting long position y_t in a futures contract, corresponding to at least one unit of the index at *T'*. The SPXW option matures at *T* and the futures at some value *T'*>*T*, the nearest futures maturity time. Current time *t* can be up to three months ahead of *T'*, apart from the last week of a month when it is an SPX, rather than an SPXW option that is being traded. At *T'* the DPM liquidates her portfolio by maximizing a concave utility function as an IT. As an OT she adds an appropriate position in one call option at *t*, which is closed at *T*. The IT portfolio is being revised along the path from *t* to *T'*, with transaction costs 1+*k* for additions to the long and 1-*k* to the short positions.

In continuous time continuous IT portfolio revisions in the presence of frictions are infeasible. As Constantinides showed in his seminal 1979 article, there exists a no trade (NT) zone for the IT investor, with trade occurring only when the risky asset dynamics bring the value of the risky asset holdings outside the NT zone. Analytical expressions for the derivation of this NT zone and the corresponding optimal IT portfolio policy for a finite horizon *T'* are available only for CRRA utilities and diffusion or jump diffusion asset dynamics, with a numerical algorithm presented in Czerwonko and Perrakis (2016). If $V(x_t, y_t, t)$ denotes the MM value function and $w_{T'}$ the terminal wealth, then we have

$$V(x_{T'}, y_{T'}, T') = U(w_{T'}) = U(x_{T'} + (1-k) y_{T'}) = w_{T'}^{1-\alpha} / (1-\alpha), \quad (4.1)$$

where α is the coefficient of relative risk aversion (RRA). As shown by Constantinides (1979), such a utility function results in a value function that is homogenous of degree α.

As noted briefly in Section 2 when we omitted the dividend rate in formulating the index return's Q-dynamics, hedging the DPM position with futures is equivalent to setting $\frac{dI_t}{I_t} = \frac{dF_t}{F_t} + \alpha_t dt$ plus a random shock equal to the basis risk. To see this denote by S_t and F_t the underlying and futures prices at t and by Z_{t+1} the ex dividend return $Z_{t+1} = \frac{I_{t+1} e^{-q_t}}{I_t} = \frac{F_{t+1} e^{\psi(t,T')}}{F_t} \Rightarrow \ln(\frac{I_{t+1} e^{-q_t}}{I_t}) = \ln(\frac{F_{t+1}}{F_t}) + \psi(t,T')$, where q_t is the (assumed constant) dividend yield and $\psi(t,T')$ the basis risk, a zero mean independent error term that varies with the distance from futures maturity. At T' we have $I_{T'} = F_{T'}$ without error. It is clear from the above formulation that the dynamics of the index return can be very closely approximated by the dynamics of the futures return in our case, provided the diffusive volatility is increased to represent the basis risk. Hereafter the basis risk will be ignored in the expressions.

The market maker at time t has dollar holdings x_t in the riskless account and y_t / F_t unit index futures and increases (or decreases) the dollar holdings in the index futures position from y_t to $y_t' = y_t + \upsilon_t$ by decreasing (or increasing) the riskless account from x_t to $x_t' = x_t - \upsilon_t - \max\left[k\upsilon_t, -k\upsilon_t\right]$. The dynamics of x_t and y_t are

$$\begin{aligned} x_{t+1} &= \left\{x_t - \upsilon_t - \max\left[k\upsilon_t, -k\upsilon_t\right]\right\} R, \quad t \le T'-1 \\ y_{t+1} &= \left(y_t + \upsilon_t\right) \frac{F_{t+1}}{F_t}, \quad t \le T'-1. \end{aligned} \quad (4.2)$$

At each date t $(t \le T'-1)$, the market maker maximizes her expected utility of terminal consumption in the form of the value function $V(x_t, y_t, t)$ of her derived utility:

$$\max_v E_t \left[V\left([x_t - v - \max(k_1 v, -k_2 v)]R, (y_t + v)\frac{F_{t+1}}{F_t}, t+1 \right) \right], \ t \le T'-1, \quad (4.3)$$

with the terminal condition $V(x_{T'}, y_{T'}, T') = U(w_{T'}) = U(x_{T'} + y_{T'})$, when all the holdings are converted into cash to form the terminal wealth $w_{T'}$ when the futures positions are liquidated. The homogeneity property is the key to the numerical approach to the recursive derivation of the value function $V(x_t, y_t, t)$, which is defined by its no trade (NT) region. This derivation was done numerically in Czerwonko and Perrakis (2016, equations A.8-A.9) and is reproduced briefly in the appendix. The numerical approach was shown to converge efficiently to a continuous time limit when the asset dynamics represented by the distribution of the index futures return $\frac{F_{t+1}}{F_t} \equiv z_{t+1}$ converged to diffusion or to jump diffusion. In what follows we shall assume that the DPM stays in the NT zone for the entire period to option expiration, which for SPXW options is a very good approximation, given that the costs for restructuring to the nearest NT boundary are very low. In such a case at option expiration *T* the value function at *t* is $V(x_t, y_t, t) = E_t[V(x_t R, y_t z_T, T)]$.

While the DPM is a licensed monopolist, she is vulnerable to competition from other players who may make quotes within the bid-ask spread, a fact that she needs to take into account in setting her quotes, for which she has informational advantages. Suppose that a would-be competitor wants to adopt a long position in *N* SPXW options with strike price K and let $J(x_t, y_t, t)$ denote her value function. The reservation purchase price C of the option is defined as the value that leaves the would-be competitor indifferent with respect to buying the option or not *given* the size *N* of the quote, in other words the lowest value of C satisfying the equation below.

$$V(x_t, y_t, t) = J(x_t - NC, y_t, t) = E_t[V(x_t - NC)R + N(F_T - K)^+, y_T, T)] \ . \tag{4.4}$$

An equivalent expression also exists for the reservation write price. Alternatively, it is possible to fix C and find the largest quote size N, long or short, that is consistent with a quoted ask or bid price that she wants to compete with.

This problem was formulated and solved theoretically and numerically by Czerwonko and Perrakis and is reported in Perrakis (2019, pp. 233-239), and the various steps are shown in the appendix. They show that any would-be liquidity provider with a given RRA, initial wealth and quote depth of N is able to provide the following limiting quotes

$$\begin{array}{ll} C_b = \max\{C\}, \text{ such that} & C_a = \min\{C\}, \text{ such that} \\ J_b\left(x_t - NC, y_t, t\right) = V\left(x_t, y_t, t\right), & J_a\left(x_t + NC, y_t, t\right) = V\left(x_t, y_t, t\right) \end{array} . \tag{4.5}$$

The following important theoretical results were also shown.

Lemma 1: For any would-be liquidity provider and for any depth of quote increasing the initial wealth yields a lower (higher) ask (bid) option quote.

Lemma 2: For any would-be liquidity provider, and under the assumption that the provider is marginal in the market for a single option and does not alter her IT [portfolio, a reduction in the depth of the quote yields a lower (higher) ask (bid) option quote.

In other words, the first result shows that there are clear wealth-related economies of scale in providing liquidity. The second result shows also equally clearly the importance for the designated DPM of estimating the demand functions at the beginning of this note. The DPM has informational advantages vis-à-vis would be competitors and it will be assumed that the reactions of such competitors are imbedded in the demand estimates.

Suppose now that the designated DPM is as described above, with fully hedged demand estimates of $N_s = D_s(C_b, C_a, t)$ and a net short exposure of $N_e(C_b, C_a, t) = D_s(C_b, C_a, t) - D_l(C_b, C_a, t)$, increasing in both arguments. Define

$$\hat{x}_t = x_t + N_l(C_a - C_b) - N_e C_b, \;\; N_l = D_l(C_b, C_a, t), \;\; N_l + N_e = D_s(C_b, C_a, t). \tag{4.6}$$

Then solve the following problem for the market maker:

$$\begin{aligned} &Max_{C_a, C_b} J_b\left(\hat{x}_t, y_t, t\right) \text{ subject to (4.6) and} \\ &J_b\left(\hat{x}_t, y_t, t\right) \geq V\left(\hat{x}_t, y_t, t\right), \;\; J_b\left(\hat{x}_T, y_T, T\right) = V(\hat{x}_T + N_e F_t (z_T - m_t)^+, y_T, T). \\ &\hat{x}_T + y_T(1-k) + N_l F_t (z_T - m_t)^+ > 0, \; \hat{x}_T + y_T(1+k) + N_e F_t (z_T - m_t)^+ > 0 \end{aligned} \tag{4.7}$$

In (4.7) we set $m_t = \dfrac{K}{F_t}$

Under the assumption that the market maker stays in the NT zone during the period from *t* to *T* we can express the first order conditions (FOC) for the problem (4.7). Since obviously $J_b\left(\hat{x}_t, y_t, t\right) = E_t[V(\hat{x}_T + N_e F_t (z_T - m_t)^+, y_T, T)] = E_t[V(R(x_t + N_l(C_a - C_b) - N_e C_b) + N_e F_t (z_T - m_t)^+, y_T, T)]$, we have from (4.6) and (4.7)

$$\begin{aligned} &E_t[V_x](R(N_l + (C_a - C_b)\frac{\partial N_l}{\partial C_a} - \frac{\partial N_e}{\partial C_a} C_b) + \frac{\partial N_e}{\partial C_a} F_t E_t[V_x (z_T - m_t)^+] = 0 \\ &E_t[V_x](-N_l R + R(C_a - C_b)\frac{\partial N_l}{\partial C_b} - N_e R - \frac{\partial N_e}{\partial C_b} R C_b) + \frac{\partial N_e}{\partial C_b} F_t E_t[V_x (Z_T - m_t)^+] = 0 \end{aligned} \; . \tag{4.8}$$

Relations (4.8) simplify into

$$
\begin{aligned}
F_t \frac{E_t[V_x(z_T - m_t)^+]}{RE_t[V_x]} &= \frac{1}{\frac{\partial N_e}{\partial C_a}}[-N_l - (C_a - C_b)\frac{\partial N_l}{\partial C_a} + \frac{\partial N_e}{\partial C_a}C_b] \\
F_t \frac{E_t[V_x(z_T - m_t)^+]}{RE_t[V_x]} &= \frac{1}{\frac{\partial N_e}{\partial C_b}}[N_l - (C_a - C_b)\frac{\partial N_l}{\partial C_b} + N_e + \frac{\partial N_e}{\partial C_b}C_b]
\end{aligned}
. \tag{4.9}
$$

If $C_a = C_b \equiv C$ then (4.9) simplifies to

$$
F_t \frac{E_t[V_x(Z_T - m_t)^+]}{RE_t[V_x]} = \frac{1}{\frac{\partial N_e}{\partial C}}[N_e + \frac{\partial N_e}{\partial C}C] > C . \tag{4.10}
$$

Under competitive conditions, assuming equal wealth and RRA for all investors and with the OT investor in the NT zone, the LHS of (4.10) is equal to the zero bid-ask spread call value C. Alternatively, the term $\frac{V_x}{E_t[V_x]}$ has the interpretation of a pricing kernel when aggregated over all investors and corresponds to the competitive price. Monopoly power raises its price above that by the elasticity of the supply curve $N_e(C,t)$. Obviously, a positive spread raises the value even further above the value prevailing under competitive conditions in (4.10).

On the other hand, the frictionless SD bounds are independent of demand conditions and hold for all types of intermediate market structure without frictions. The all-important SD upper bound imposes limits on the RRA's of the IT investors participating in the option market and is consistent with a monopolistic liquidity provider with asymmetric information who may face competition from small traders. This upper bound is also sharply higher for deep OTM put options. It is, therefore, not surprising that in our tables these deep OTM put options' quoted prices are largely consistent with the SD upper bound, with more than 80% of the put contracts in these categories having the bid-ask midpoint lying within the bid-ask spread. This is in sharp contrast with the also highly liquid ATM category of the moneyness ratio $0.98 < \frac{K}{F_t} < 1.02$, where for a clear majority of the quoted contracts there is no overlap between the SD bounds and the quoted spread. Since Jouini and Kallal (1995, Theorem 3.2, p. 188) showed that a "correctly" priced frictionless option market equilibrium *must* lie within the bid-ask spread, the above results imply that the bid-ask midpoint in the ATM portion of the option quotes is not an acceptable proxy for the frictionless equilibrium, while it is for the deep OTM options! These results contradict the practice in most empirical option research studies of fitting the *Q*-dynamics' parameters to the bid-ask midpoint of the entire cross section. They also provide a rigorous alternative to the suggestions of AFT "to include the priced tail risk as a genuine state variable, distinct from market volatility, in parametric models."

Relations (4.4)-(4.10) predict that for both calls and puts increases in the net exposure will ceteris paribus increase the distance between the spread midpoint and the SD option upper bound and, thus, increase the mispricing in frictionless trading. They also predict that relative tightness of the quotes will be associated with smaller depths of the quotes in order to deter entry from 'small" players who will undercut the quotes with even smaller depths. The effects on the spread of increases in the relative sizes of the hedged and unhedged portions of the option market will depend on the particular case and are not clear cut, but they will certainly be affected. These are all testable predictions, and they also allow model verification.

V. Conclusions

In this paper we have shown that the SPXW option quotes do not conform even approximately to the frictionless option market format, which has been assumed to hold in the overwhelming majority of empirical index option market research. In particular, by applying the SA approach for frictionless markets we showed that the constant volatility jump diffusion asset dynamics for the index returns that are suitable for such short maturities set stringent limits for the risk neutral valuation of the admissible option values. These limits are violated in a major way by the observed midpoints of the option quotes in a very large portion of the time series of our data. These violations correspond to tradable anomalies in the *frictionless* market, which can be tested rigorously under such conditions. The test overwhelmingly reject the null of non-dominance of OT over IT for both calls and puts, and conclude that a large proportion of the prices in the observed cross sections are mispriced.

Our results confirm also rigorously the AFT findings of inconsistency between the left tail OTM puts and the rest of the frictionless cross section, by an entirely different approach. They also provide an empirical justification of the Jouini-Kallal (1995) theory, that the frictionless risk neutral prices of the options are "correct" if and only if they lie within the bid-ask spread of the market with frictions. Last, they justify the empirical failure of the Ross (2015) Recovery Theorem, that attempted to reconcile the disconnect in left tail risk between the *P*- and *Q*-dynamics, but ended up with the disconnect unchanged: you cannot recover what is not there, and the "correct" *Q*-dynamics emphatically did not lay where most empirical option research was searching, at the observed bid-ask spreads of the option market.

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Table 1: Summary Statistics Weeklys

K/S	.57-.86	.86-.90	.90-.93	.93-.96	.96-.98	.98-1.0	1.0-1.02	1.02-1.04	1.04-1.06	1.06-1.20	Total
Panel A: Calls											
# Contracts	126	155	314	1,271	1,869	2,038	2,258	2,475	1,469	731	12,706
% Contracts	1%	1%	2%	10%	15%	16%	18%	19%	12%	6%	100%
Avg IV	70.2%	45.8%	34.0%	24.3%	17.9%	14.6%	11.8%	12.2%	15.7%	22.4%	16.2%
Avg Moneyness	0.825	0.883	0.919	0.949	0.971	0.990	1.011	1.030	1.049	1.077	1.001
Avg Mid-Quotes	404.0	263.3	178.2	112.6	65.6	29.6	6.0	1.1	0.4	0.2	38.6
Avg Delta	0.98	0.97	0.96	0.94	0.88	0.70	0.25	0.05	0.02	0.01	0.39
Avg Vega	16.9	17.6	23.1	32.9	52.1	88.3	76.3	23.1	9.7	5.7	46.3
Relative Spread	2.7%	3.1%	3.6%	4.2%	4.9%	4.9%	10.1%	39.6%	60.2%	67.8%	22.5%
SD Spread	0.1%	0.4%	0.8%	1.6%	3.2%	8.0%	49.9%	125.0%	149.0%	135.6%	60.2%
Sum Net Demand	-0.1	-0.1	-2	12	29	-843	-148	-265	-156	-129	-745
Sum Open Interest	4	9	106	825	2,672	6,619	11,739	9,048	3,800	1,937	36,760
Sum Volume	1	1	7	66	156	1,321	6,246	4,163	1,495	789	14,244
Panel B: Puts											
# Contracts	3,721	3,997	3,719	3,983	2,604	2,111	2,043	1,162	444	277	24,061
% Contracts	15%	17%	15%	17%	11%	9%	8%	5%	2%	1%	100%
Avg IV	53.6%	36.9%	29.3%	22.9%	17.8%	14.8%	12.6%	16.0%	23.1%	33.7%	28.9%
Avg Moneyness	0.818	0.881	0.915	0.945	0.970	0.989	1.010	1.028	1.049	1.078	0.930
Avg Mid-Quotes	0.2	0.3	0.6	1.3	3.2	8.4	27.6	61.6	103.1	167.8	10.62
Avg Delta	0.00	-0.01	-0.02	-0.04	-0.10	-0.29	-0.72	-0.89	-0.93	-0.94	-0.16
Avg Vega	3.1	5.7	10.5	21.9	47.0	88.0	84.1	47.9	36.4	34.3	29.3
Relative Spread	60.3%	50.9%	34.8%	20.2%	10.4%	5.8%	7.1%	6.4%	4.8%	3.4%	29.2%
SD Spread	190.4%	181.0%	169.0%	153.9%	128.8%	64.5%	3.7%	0.5%	0.2%	0.1%	131.0%
Sum Net Demand	251.9	531.0	152	-395	-320	-9	-120	-15	-7	8	75,347
Sum Open Interest	10,414	12,796	12,740	16,482	12,251	8,596	3,235	1,352	512	378	78,756
Sum Volume	6,454	6,042	5,153	6,543	6,876	5,664	1,272	132	55	27	38,219

This table presents summary statistics for the merged CBOE time-stamped End-of-Week Weekly option quotes (EoW Weeklys) with trading volume from Market Data Express Open/Close database. Option quotes are at 3:00 PM from 20091106 to 20190719 for the total of 380 cross sections. "Relative Spread" ("SD Spread") is the average bid-ask (upper and lower bound) spreads in proportion of mid-quotes (mid-bounds) in each moeyness category. "Net Demand" (in thousands) is the net buy and sell by end-users. "Open Interest," "Volume," (in thousands) "IV," "Delta," and "Vega" are from OptionMetrics end-of-day data.

Table 2: Weeklys Mispricing with respect to SD Bounds

K/S	.57-.86	.86-.90	.90-.93	.93-.96	.96-.98	.98-1.0	1.0-1.02	1.02-1.04	1.04-1.06	1.06-1.20
Mid-quotes within Bounds	16	70	162	742	1,028	284	230	1,081	894	249
	13%	45%	52%	58%	55%	14%	10%	44%	61%	34%
Overpriced Contracts	0	0	6	111	421	1,495	1,810	936	250	186
	0%	0%	2%	9%	23%	73%	80%	38%	17%	25%
Cross-sections >1 Traded Option	57	93	158	330	358	344	363	370	328	169
	15%	24%	42%	87%	94%	91%	96%	97%	86%	44%
Cross-sections >1 Overpriced	0	0	6	43	129	293	313	215	73	49
	0%	0%	4%	13%	36%	85%	86%	58%	22%	29%
Cross-sections >50% Overpriced	0	0	4	36	81	235	271	134	47	29
	0%	0%	3%	11%	23%	68%	75%	36%	14%	17%
Cross-sections 100% Overpriced	0	0	4	31	63	168	223	77	33	21
	0%	0%	3%	9%	18%	49%	61%	21%	10%	12%
Panel B: Puts										
Mid-quotes within Bounds	2,396	3,701	3,239	3,183	1,520	299	157	140	37	8
	64%	93%	87%	80%	58%	14%	8%	12%	8%	3%
Overpriced Contracts	1,180	242	320	667	1,028	1,755	1,239	159	21	1
	32%	6%	9%	17%	39%	83%	61%	14%	5%	0%
Cross-sections >1 Traded Option	327	374	376	377	369	344	360	328	156	60
	86%	98%	99%	99%	97%	91%	95%	86%	41%	16%
Cross-sections >1 Overpriced	108	30	47	100	220	306	277	49	6	1
	33%	8%	13%	27%	60%	89%	77%	15%	4%	2%
Cross-sections >50% Overpriced	37	24	39	70	151	278	201	23	4	0
	11%	6%	10%	19%	41%	81%	56%	7%	3%	0%
Cross-sections 100% Overpriced	23	23	31	57	116	214	85	16	1	0
	7%	6%	8%	15%	31%	62%	24%	5%	1%	0%

This table presents statistics about Weeklys mispricing. First (second) row shows the number (percentage) of contracts where mid quotes are within the SD bounds. Third (fourth) row shows the number (percentage) of overpriced contracts. Fifth (sixth) row shows the number (percentage) of cross sections with at least one traded contracts. The last six rows show the number and percentage of cross sections where at least 1 contract, 50% of contracts, and all contracts in that cross section is overpriced.

Table 3: Davidson-Duclos Test for the Second Order Stochastic Dominance

Difference in Mean	$\mu_{IT} \geq \mu_{OT}$	OT Non-dominance (No Trimming)	OT Non-dominance (5% Trimming)	OT Non-dominance (10% Trimming)
Weeklys - Calls				
0.0005	0.0118	0.0100	0.0000	0.0000
Weeklys - Puts				
0.0007	0.0030	0.0000	0.0000	0.0000

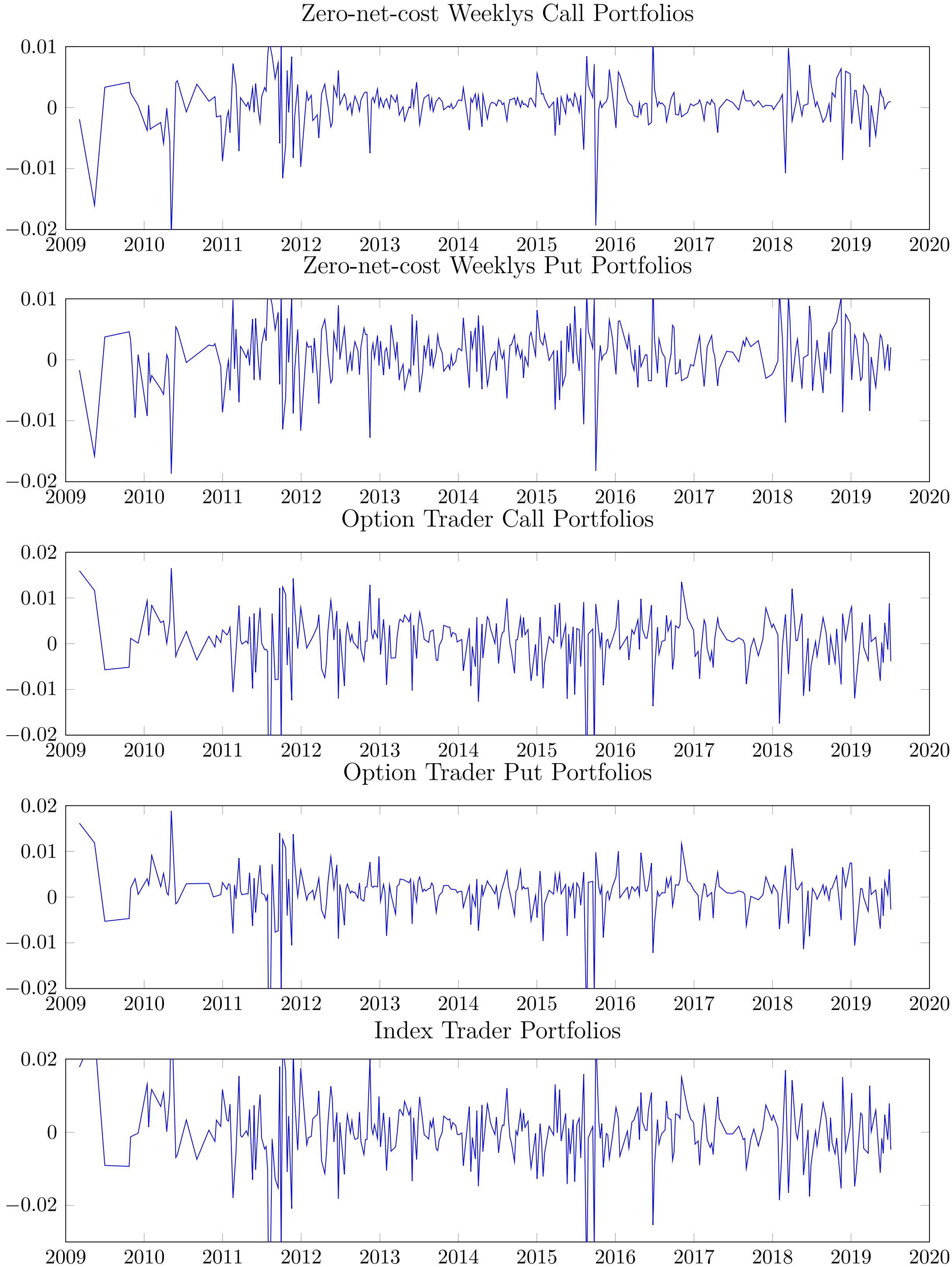
This table presents p-values for Davidson-Duclos (2007) second order stochastic dominance test for paired (correlated) outcomes. The first column reports p-values for the difference in means of one-day index trader (IT) portfolio returns and an option trader (OT) portfolio returns. The second column reports p-values for the bootstrap test of the null that mean returns of an IT portfolio is higher than that of an OT portfolio. The last three columns report p-values for the null of non-dominance of OT portfolio over IT portfolio, with no trimming in the right tail (third column), 5% trimming in the right tail (fourth column), and 10% trimming in the right tail (fifth column). In all panels, the first row results are obtained where OT short overpriced option at bid quote and close her position at SD upper bounds.

Table 4: Option Trader Portfolio Returns

	Open Bid / Close UB	Open Bid / Close Ask	Open Mid-Quotes / Close Mid-Quotes
Weeklys - Calls			
Average Return	0.1482	0.0012	0.1062
Standard Deviation	0.1055	0.1037	0.1062
Weeklys - Puts			
Average Return	0.2391	0.0430	0.1476
Standard Deviation	0.0903	0.0955	0.0942

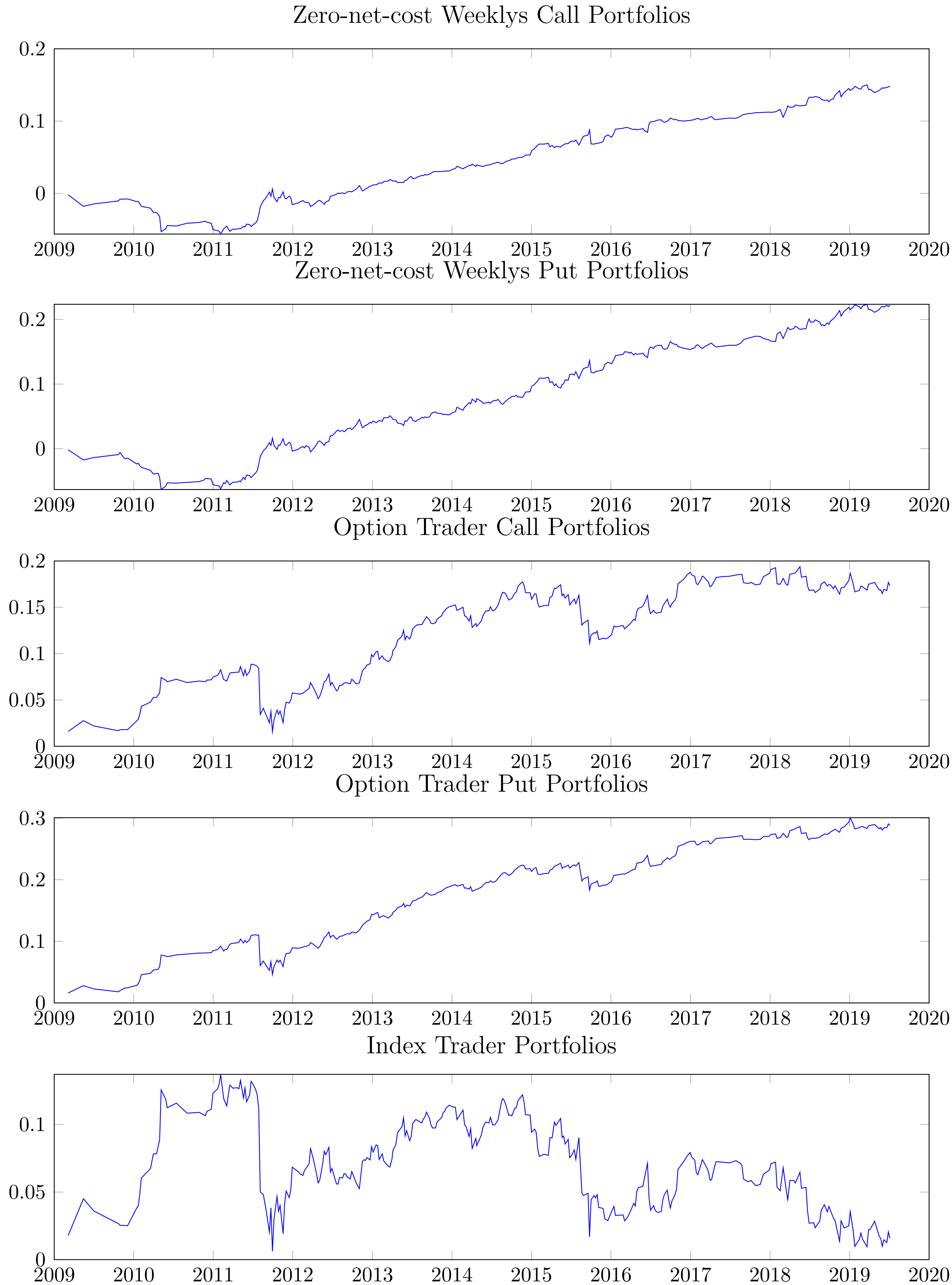
The table reports arithmetic average returns (annualized) and standard deviations for an option trader (OT) portfolio. The main results are obtained when an OT writes an overpriced option at its bid quote and closes her position at the SD upper bound. For robustness, we also report average returns when an OT portfolio is set by writing an overpriced option at its bid quote and closing the position at the ask quote. The robustness of an OT portfolio returns also examined when the portfolio is set and closed at mid bid and ask quotes.

Figure 1: 1-day Log-Return



The first (second) panel shows the weighted average zero-net-cost call (put) option (Weeklys) portfolio returns (1-day return) in each cross section. The weight of each option portfolio in cross sectional portfolio is relative to the SD bounds violation for each option. The third (fourth) panel plots OT call (put) portfolio returns in each cross section. The last panel plots one-day returns of the IT portfolio in each cross section.

Figure 2: Cumulative 1-day Log-Return



The first (second) panel shows the cumulative zero-net-cost call (put) option (Weeklys) portfolio returns (1-day return) in each cross section. The weight of each option portfolio in cross section is relative to the SD bounds violation for each option. The third (fourth) panel plots cumulative OT call (put) portfolio returns in each cross section. The last panel plots cumulative returns of the IT portfolio in each cross-section.